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Optimal Portfolio Liquidation with Distress Risk

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We analyze the problem of an investor who needs to unwind a portfolio in the face of recurring and uncertain liquidity needs, with a model that accounts for both permanent and temporary price impact of trading. We first show that a risk-neutral investor who myopically deleverages his position to meet an immediate need for cash always prefers to sell more liquid assets. If the investor faces the possibility of a downstream shock, however, the solution differs in several important ways. If the ensuing shock is sufficiently large, the nonmyopic investor unwinds positions more than immediately necessary and, all else being equal, prefers to retain more of the assets with low temporary price impact in order to hedge against possible distress. More generally, optimal liquidation involves selling strictly more of the assets with a lower ratio of permanent to temporary impact, even if these assets are relatively illiquid. The results suggest that properly accounting for the possibility of future shocks should play a role in managing large portfolios.

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1. Introduction

Following the crisis that surrounded the downfall of Long-Term Capital Management in 1998, Myron Scholes (2000) raised the following problem: How should an investment manager unwind a portfolio when faced with present and possible future liquidity needs? This problem is ubiquitous during unstable financial periods. For example, in the recent financial crisis, banks incurred large losses during the forced contraction of their balance sheets as access to short-term financing through repo markets dried up (e.g., Adrian and Shin 2008, Brunnermeier 2009). A systemic deleveraging process propagated through the banking sector, in which careful liquidation became crucial to preserving wealth and surviving the crisis.

The key question here is which assets should be sold to meet short-term obligations, keeping in mind the uncertainty regarding liquidity needs in the future. This uncertainty, which we informally term distress risk, may arise because of unforeseen equity withdrawals or higher cash requirements to fund other areas of the business, possibly in conjunction with less favorable funding conditions such as tighter margin constraints. This problem is distinct from a related problem that has been extensively analyzed in the past. Previous work has focused on the optimal way to liquidate a single asset, either as a monopolist (e.g., Bertsimas and Lo 1998, Huberman and Stanzl 2005, Schied and Schöneborn 2009) or against selling pressure (e.g., Brunnermeier and Pedersen 2005, Carlin et al. 2007).[1] In these papers, a trader needs to sell a given asset for exogenous reasons. In the problem addressed in this paper, the trader needs to generate cash and reduce leverage and chooses which assets to sell. The solution to this problem has fundamentally different economic implications.

We develop a two-period model where, in each period, the net cost of trading and the price impact of trading on the market value of the assets is based on a continuous-time market. In keeping with the optimal liquidation literature, we take the price impact of trading the assets as a primitive rather than derive a general equilibrium in the market. A single, risk-neutral investor holds a portfolio of assets, where the market price of each asset is impacted by trading depending on its permanent and temporary components of liquidity. The permanent component of liquidity is the change in the asset's price that depends on the cumulative amount traded, and is independent of the rate at which the asset is traded.

[1] See also Vayanos (1998), Almgren and Chriss (1999, 2000), Fedyk (2001), DeMarzo and Urošević (2006), Oehmke (2008), and Chu et al. (2009).

The temporary component of liquidity measures the instantaneous, reversible price pressure that results from trading. Partitioning liquidity into these components has been shown to be empirically robust and important (e.g., Sadka 2006; see Carlin et al. 2007 for further discussion).

We begin by analyzing a single-period version of the problem in which the trader does not consider future needs for liquidity—that is, there is no possibility of needing to liquidate assets in the second period. The investor faces a limit on leverage (a margin constraint), and experiences an urgent need for liquidity. We call this scenario *myopic deleveraging*. We find that, for a given portfolio and price impact parameters, the investor optimally sells more of the assets that are liquid to meet pending obligations.

This result changes when we consider a two-period problem as follows. In the first period, the investor is required to unwind part of the portfolio to reduce leverage. Subsequently, with some probability, the investor may experience another liquidity shock and be required to further unwind the portfolio in the second period. If no further distress occurs, trading ends. However, if the investor suffers further distress, the problem faced in the second period is similar in nature to the single-period case. The investor acts in accordance with given assessments of the probability and size of a second-period shock.

A central question of interest is: How do the liquidity characteristics of the assets affect the trader's ability to hedge distress risk? A trade-off arises in the first period of the two-period problem. Selling the more liquid assets first will limit the immediate loss in value; however, the resulting portfolio will be more vulnerable to a continued shock in future periods. Selling the less liquid assets first will result in a portfolio that is more robust to a continued adverse environment; however, this can result in possibly unnecessary loss in value if there is no subsequent shock.

The solution in the two-period model is qualitatively different from the myopic deleveraging case in several ways. In the case of myopic deleveraging, the investor will only trade just enough to meet the margin constraint; because trading is costly, there is no benefit to trading any more than necessary. This is not the case in the two-period model. When the expected second-period shock is large enough, the investor will always want to trade away from the margin in the first period. In doing so, the investor retains cash to protect against a future shock.

We also find that, for large shocks, the amount sold of each asset is monotonically decreasing (across assets) in the ratio of permanent to temporary price impact. This leads to a key difference relative to the single-period case, which suggests that the temporary component of liquidity is central to portfolio management in the two-period model. If the expected need for liquidity is small, the investor behaves in a similar way as in the one-period problem. However, when the expected need for liquidity is large, the investor holds on to assets with a low temporary impact of trading and, all else being equal, sells assets that are less liquid on this dimension. The same does not hold for the permanent price impact. No matter how large the size of the second-period shock, all else equal, the investor always favors selling more of the assets with a low permanent price impact of trading in the first period.

This sheds light on the nature of the practical solution to the portfolio liquidation problem. Assets with concentrated ownership or those with a high degree of asymmetric information (i.e., those with high permanent price impact) will not be prioritized for liquidation when an investor experiences a recurrent need for liquidity. Assets that are heavily traded, where there are many opportunities to access counterparties, may or may not be liquidated early. If the expected need for liquidity is small, the investor optimally sells these securities to meet early obligations. However, if the expected need for liquidity is large, the investor will retain these assets, preserving the option to sell them in the future. Whether large investors do in fact take such considerations into account during crisis remains an open empirical question. Although our analysis is supported by findings in Manconi et al. (2010), more work is required to validate our results empirically.[2]

The analysis in this paper may have relevance in assessing capital requirements as buffers against financial risks associated with portfolios. As one example, consider value at risk (VaR), which measures the possible loss of a position over a target horizon that will not be exceeded with a given probability. Because the horizon is commonly set to "the longest period needed for an orderly portfolio liquidation" (Jorion 2007, p. 20), conventional calculations

[2] Manconi et al. (2010) study the transmission of the 2007–2008 crisis from the market for securitized bonds into the corporate bond market. They show that, at the onset of the crisis, investors retained the (now illiquid) securitized bonds and chose instead to liquidate the corporate bonds to meet their funding needs. They also find that investors expected to be more exposed or more severely constrained due to having shorter investment horizons, liquidate relatively more of low-grade corporate bonds (assumed to have higher transaction costs) rather than high-grade corporate bonds. Based on this, they suggest that more constrained investors preferred to liquidate assets with lower permanent price impact and higher temporary price impact. Although this is consistent with our findings, more empirical work is needed. In particular, it would be important to develop an empirical ranking of the low-grade corporate bonds and securitized bonds in terms of temporary and permanent price impact along the lines of Sadka (2006).

of VaR may underestimate financial risk when liquidation is forced to occur over shorter time intervals. As the liquidity characteristics of particular assets affect the value at risk during forced liquidation, the analysis in this paper might be used to refine VaR or related measures of risk that are used in assessing capital requirements.

Finally, our analysis adds to a rather large literature on optimal liquidation, which has focused on the case of a single asset. One exception is Duffie and Ziegler (2003), who numerically investigate the trade-off between selling off an illiquid asset to keep a "cushion of liquid assets," and selling a liquid asset to maximize short-term portfolio value. Illiquidity is modeled as linear transaction costs, and price impact of trading is not considered (they note, however, that this may be a central concern for large investors). Our paper considers the effect of price impact and, albeit with a considerably simpler model of uncertainty, provides an analytical derivation of structural properties of the optimal solution for a portfolio with any number of assets.

We present the one-period model in §2, the two-period model in §3, and conclude in §4. All proofs are in the appendix.

2. One Period: Myopic Deleveraging

2.1. Price and Trading Model

Consider a single risk-neutral investor who trades a portfolio of n assets in continuous time over a finite horizon.[3] At any time $t \in [0, \tau]$, $Y_t \in \mathbb{R}^n$ is the rate at which the investor trades the assets. The investor's holdings are denoted by $X_t \in \mathbb{R}^n$, where $X_t = x_0 + \int_0^t Y_s\, ds$. We will generally assume $x_0 > 0$. We assume that Y_t is an L^2-function.

The prices of the assets at time t are given by $P_t \in \mathbb{R}^n$, which is determined by

$$P_t = q + \Gamma X_t + \Lambda Y_t. \qquad (1)$$

This is a multidimensional version of the pricing equation used in Carlin et al. (2007). A similar pricing relationship was derived in equilibrium by Gennotte and Kyle (1991) and also by Vayanos (1998). Pritsker (2004) and Huberman and Stanzl (2004) obtain similar relationships. We characterize an investor's optimal trading behavior as a function of current and possible future liquidity needs, taking Equation (1) as given for the price equilibrium in a market in which there is price impact of trading (and otherwise independent of the investor's optimal strategy).

The expression in (1) has three parts. The first term, $q \in \mathbb{R}^n$, specifies the intercept of the linear model, that is, the equilibrium prices that arise when the investor does not hold any assets and is not trading. For each asset in the portfolio, q measures the present value of underlying future cash flows. By keeping q independent of time (and fixed across periods in the two-period model), we are able to focus on the consequences of uncertainty about future liquidity shocks, while keeping the problem analytically tractable. We believe this assumption is sensible for small τ and large shocks, and also for considering a particular trader's idiosyncratic need to liquidate a position.

The second and third terms partition the price impact of trading into permanent and temporary components. The permanent component measures the change in price that is independent of the rate at which any of the assets are traded. This impact is likely to be high when the amount of asymmetric information associated with an asset is high or ownership of the asset in the market is concentrated. The temporary component measures the instantaneous, reversible price pressure that results from trading, which can also be interpreted as a transaction cost. This component is likely to be high when the asset is thinly traded or there is a paucity of readily available counterparties in the market. Both $\Gamma \in \mathbb{R}^{n\times n}$ and $\Lambda \in \mathbb{R}^{n\times n}$ are diagonal matrices in which diagonal entries i (γ_i and λ_i) are the price impacts of asset i. We assume that these price impact parameters are positive, implying that the matrices Λ and Γ are both positive definite ($x'\Gamma x > 0$ and $x'\Lambda x > 0$ for all $x \in \mathbb{R}^n$ such that $x \neq 0$).[4]

Partitioning the price impact of trading into permanent and temporary components is justified on empirical grounds. Kraus and Stoll (1972), Holthausen et al. (1990), and Madhavan and Cheng (1997) find large permanent and temporary effects for block trades on the New York Stock Exchange (NYSE). Madhavan and Cheng (1997) estimate these effects for block trades exceeding 10,000 shares in the "downstairs" and "upstairs" markets at the NYSE. Sadka (2006) finds that the correlation between the temporary and permanent components of price impact to be approximately 0.28, which suggests a significant variation in the ratio of permanent to temporary price impact. For a complete discussion, see Carlin et al. (2007).

[3] Risk neutrality is not an altogether unreasonable approximation if the investor is, say, a large bank, or a highly leveraged hedge fund. Nonetheless, as suggested by a numerical example in §3, risk neutrality is not required for the solution to exhibit the structural properties derived in the paper.

[4] One can easily incorporate off-diagonal entries in Λ and Γ and still solve numerically the resulting optimization problems that we will formulate. Doing this, we find that much of the qualitative behavior that we establish here, such as preemptive deleveraging, still holds. However, we can then no longer derive analytical expressions for the optimal trades.

We denote the initial and final positions by $x_0 = X_0$ and $x_1 = X_\tau$, and the cumulative trade by $y_1 = x_1 - x_0$. Prior to trading, the asset prices are $p_0 = P_{0_-} = q + \Gamma x_0$. After trading is complete, the price is

$$p_1 = P_{\tau_+} = q + \Gamma x_1 = q + \Gamma(x_0 + y_1) = p_0 + \Gamma y_1.$$

Using the notation r' to denote the transpose of a vector r, the end-of-period assets are

$$a_1 = p_1' x_1 = (p_0 + \Gamma y_1)'(x_0 + y_1) = a_0 + (p_0 + \Gamma x_0)' y_1 + y_1' \Gamma y_1,$$

a quadratic function of y_1. The cash that is generated from trading over $[0, \tau]$ is

$$\kappa_1 = \int_0^\tau -P_t' Y_t \, dt = \int_0^\tau -\left(p_0 + \Gamma \int_0^t Y_s \, ds + \Lambda Y_t\right)' Y_t \, dt.$$

We assume that cash is counted directly against liabilities (e.g., for the satisfaction of margin constraints). Denoting the initial liabilities by l_0, the liabilities at time τ can then be written as $l_1 = l_0 - \kappa_1$. Consider an investor who wishes to maximize the equity of the portfolio at the end of trading, which is calculated by simple accounting to be $e_1 = a_1 - l_1$. The optimal execution schedule is a constant trading rate, that is

$$Y_t^* = \frac{1}{\tau} y_1, \quad t \in [0, \tau].$$

For any given set of trades y_1 and resulting final prices p_1 and assets a_1, the concavity of the integrand in κ_1 leads the trader to smooth trades over time. This allows us to simply focus on how much to liquidate of each asset. Because trading occurs at a constant rate for each asset, the end-of-period liabilities may be computed as

$$l_1 = l_0 + (p_0 + \Lambda y_1 + \tfrac{1}{2}\Gamma y_1)' y_1 = l_0 + p_0' y_1 + y_1'(\Lambda + \tfrac{1}{2}\Gamma) y_1,$$

where, for notational ease, we suppress the period duration τ by a scaling of Λ. We can use this to express the equity e_1 as

$$e_1 = a_1 - l_1 = e_0 + x_0' \Gamma y_1 - y_1'(\Lambda - \tfrac{1}{2}\Gamma) y_1. \tag{2}$$

The end-of-period equity is strictly concave in the trade vector y_1 if and only if $\Lambda - \frac{1}{2}\Gamma$ is positive definite. We assume this to be the case, that is

$$\Lambda \succ \tfrac{1}{2}\Gamma, \tag{3}$$

ensuring that the trader's problem is well posed. This is a reasonable condition to ensure economic soundness of the model, as without this restriction the trader may embark on trades of infinite size and, in doing so, obtain arbitrarily large equity. From a computational standpoint, this ensures that the optimization problem involves maximization of a strictly concave objective over a bounded, convex set.[5]

2.2. Margin Constraint and Forced Deleveraging

Because of either margin requirements imposed by lenders or regulatory constraints, there is, under normal circumstances, a limit on the financial leverage that an investor can incur. Different ratios quantifying the degree to which an investor is leveraged can be found in the literature. Three commonly used ratios are liabilities over assets, assets over equity, and liabilities over equity. All three ratios are increasing in the degree of financial leverage, and are readily related to each other by $l/a = (l/e)/(l/e + 1)$ and $a/e = l/e + 1$. We specify limits on financial leverage via a bound ρ_1 on the ratio of debt to equity, that is

$$\frac{l_1}{e_1} \le \rho_1.$$

This inequality can be written as a quadratic constraint on y_1,

$$\rho_1 e_0 - l_0 + (\rho_1 \Gamma x_0 - p_0)' y_1 - y_1'(\rho_1(\Lambda - \tfrac{1}{2}\Gamma) + \Lambda + \tfrac{1}{2}\Gamma) y_1 \ge 0. \tag{4}$$

If no leverage is allowed after the trading period ($\rho_1 = 0$), this constraint is $-p_0' y_1 - y_1'(\Lambda + \frac{1}{2}\Gamma) y_1 \ge l_0$, which states that, after accounting for transaction costs, the trades must generate enough cash to cover all liabilities. If arbitrarily large leverage is permitted ($\rho_1 \to +\infty$), the constraint becomes the solvency constraint $e_1 \ge 0$.[6]

We are interested in modeling liquidity shocks that force an investor to quickly sell assets to reduce leverage or, equivalently, to generate cash to meet liabilities. In this model of a fire sale, the investor is assumed restricted from increasing positions and from short selling, which corresponds to the box constraints

$$-x_{0,i} \le y_{1,i} \le 0, \quad i = 1, \ldots, n,$$

where the additional index denotes the ith scalar component of the vectors.

[5] In practice, an investor may of course want to incorporate a number of other constraints into the problem. From the point of view of computational tractability, any modification that preserves the convexity of the problem can be easily handled. This includes constraints on position size or on trade size or any number of risk constraints in a mean-variance framework (see, e.g., Lobo et al. 2007).

[6] The condition for the constraint on leverage to be convex and bounded is that $\Lambda \succ ((\rho_1 - 1)/(2(\rho_1 + 1)))\Gamma$. (Note that if $\rho_1 \le 1$, the constraint is convex for any Λ and Γ such that $\Lambda \succ 0$ and $\Gamma \succ 0$.) This condition is implied by (3) and is therefore automatically ensured in our framework. It is a less restrictive assumption than (3) so that the problem may be bounded for some objective functions that are not concave. Though maximization of a nonconcave function over a convex set is in general an intractable problem, in this case we can still solve the problem by using a result from convex analysis known as the *S-lemma* (see, e.g., Pólik and Terlaky 2007), a quadratic analog to the Farkas lemma. If condition (3) is not satisfied, however, an unrestricted trader may still improve equity arbitrarily by repeated trading.

In what follows, we consider the problem in which the investor maximizes the value of his equity subject to both a margin constraint and the box constraints. The problem of maximizing equity subject to the box constraints and a requirement to generate a fixed amount of cash yields qualitatively similar solutions. We remind the reader that, in everything that follows, we will be considering sales of assets, which correspond to negative values in the trades vector. Thus, $y_{1,i} < y_{1,j}$ should be interpreted as meaning we liquidate *more* of asset i than asset j.

A recurring assumption in our analysis in both the single- and two-period models is that, for all i, $\rho_1 \gamma_i x_{0,i} \le p_{0,i}$. This condition is not restrictive in that, if it is violated for a particular asset, selling any amount of that asset increases the investor's leverage. To see this, note that

$$\left.\frac{\partial e_1}{\partial y_{1,i}}\right|_{(y_{1,i}=0)} = \gamma_i x_{0,i},$$

$$\left.\frac{\partial l_1}{\partial y_{1,i}}\right|_{(y_{1,i}=0)} = p_{0,i}.$$

If $\rho_1 \gamma_i x_{0,i} > p_{0,i}$, sales of asset i lead to a rate of decrease of liabilities that is not large enough to compensate for the rate of decrease of net equity (this can also be seen by taking the gradient with respect to the quadratic margin constraint (4)). It can never be optimal to sell an asset that both hurts equity and increases leverage, and such an asset can therefore be excluded from the problem without loss of generality.

Result 1. *Consider the single-period deleveraging problem*

$$\begin{aligned} &\textit{maximize} \;\; e_1 \\ &\textit{subject to} \;\; l_1 \le \rho_1 e_1 \\ &\qquad\qquad -x_0 \le y_1 \le 0, \end{aligned}$$

where deleveraging is required ($l_0/e_0 > \rho_1$). *The optimal solution satisfies* $l_1/e_1 = \rho_1$ *and there exists a* $z > 0$ *such that the optimal trades are given by*

$$y^*_{1,i} = \max\left(-x_{0,i}, \min\left(0, \frac{1}{2}\frac{(1+z\rho_1)\gamma_i x_{0,i} - z p_{0,i}}{(1+z\rho_1)(\lambda_i - \frac{1}{2}\gamma_i) + z(\lambda_i + \frac{1}{2}\gamma_i)}\right)\right).$$

The optimal trade of asset i, $y^*_{1,i}$, *is increasing in* ρ_1 *and decreasing in* l_0, *but is not, in general, monotonic in* $x_{0,i}$, *in* λ_i, *nor in* γ_i.

Not surprisingly, the trader will sell off to the point where the margin constraint binds. This maximizes value in the single-period case but, as we will see in the next section, may not hold in the multiperiod setting. Trades are nonmonotonic in the price-impact parameters due to two opposing effects, which can be appreciated as follows. Consider an asset for which the price impact of trading increases, and how this changes the optimal trades. On the one hand it will be comparatively more costly to deleverage, requiring the investor to liquidate a larger share of the portfolio. On the other hand, the investor will prefer to sell less of this particular asset, and more of others. Which effect dominates as to the amount that is liquidated of the asset in question is determined by how quickly the optimal trades shift away from that asset, versus how quickly the fraction of the portfolio that needs to be liquidated increases. Simple examples can be constructed where, over a reasonable range for the price impact parameters, $y^*_{1,i}$ exhibits nonmonotonic behavior.

The next result establishes an asset ordering for myopic deleveraging.

Result 2. *For the single-period deleveraging problem, assets with low price impact are prioritized for liquidation. If two assets* i *and* j *are such that* $p_{0,i} = p_{0,j}$, $\gamma_i \le \gamma_j$, $\lambda_i \le \lambda_j$, *and* $x_{0,i} = x_{0,j}$, *then* $y^*_{1,i} \le y^*_{1,j}$.

Over assets that are otherwise identical and for which the investor has similar holdings, more of the liquid assets are sold. Note that, whereas the proofs of the monotonicity with respect to λ_i and $x_{0,i}$ are trivial from the partial derivatives of $y^*_{1,i}$ and hold without the distressed-deleveraging box constraints, this is not the case for γ_i, for which the monotonicity does not hold without the no-shorting constraint. The monotonicities in the price impact parameters imply that, in distressed sales due to short-lived shocks, traders should deleverage by selling off their most liquid holdings to generate cash or decrease their liabilities. We illustrate this with a simple four-asset example that we will later revisit.

Example 1. Consider an investor with four assets, each with price \$5, and holding one million units of each, for current holdings of $a_0 = \$20$ M. Liabilities are \$19 M, and the maximum allowable leverage is $\rho_1 = 18$, forcing the investor to partially unwind the positions.

The assets have temporary impacts of $\lambda_1 = 0.06$, $\lambda_2 = 0.055$, $\lambda_3 = 0.2$, and $\lambda_4 = 0.015$, and permanent impacts of $\gamma_1 = 0.016$, $\gamma_2 = 0.015$, $\gamma_3 = 0.006$, and $\gamma_4 = 0.006$ (with each impact parameter in units of dollars per million units traded). Thus, according to both temporary and permanent price impacts, the assets are in order, from 1 to 4, of increasing liquidity, with assets 1 and 2 considerably more illiquid than assets 3 and 4.

Solving the myopic deleveraging problem, we find the optimal trade is $y^*_{1,1} = 0$, $y^*_{1,2} = 0$, $y^*_{1,3} = -0.0851$ M, and $y^*_{1,4} = -0.1205$ M. Consistent with Result 2 and with intuition, most of the sale is of the most liquid assets (assets 3 and 4), and, in fact, the optimal solution involves selling nothing of the least liquid assets (assets 1 and 2). Asset 3 has temporary impact that is 33% higher than that of asset 4, so it is also not surprising that $y^*_{1,4} < y^*_{1,3}$. These sales result in $e^*_1 = \$0.9985$ M, so that liquidation costs the investor 15 basis points of net equity.

3. Deleveraging with Risk of Future Liquidity Shock

We now consider a situation in which the investor considers the possibility of future shocks. We first discuss the two-period model that we study and then provide some results on the nature of the optimal solution.

3.1. Model

Consider again a single investor who trades in n assets over two periods. Each period lasts a discrete amount of time τ, in which trading occurs continuously as before. Prices arise from the process in (1) and the investor is restricted to satisfy constraints on leverage at the end of both periods. Because we wish to study policies regarding deleveraging under fire-sale conditions, we restrict the investor's trades to be reductions in positions and disallow shorting. We believe this is an appropriate restriction because, around a crisis event, the investor would likely be reluctant to, or even restricted from, hastily increasing exposure.

The key difference now is that there is uncertainty during the first period about whether the investor will face the need for further liquidity during the second period. This uncertainty may arise because of unforeseen equity withdrawals or higher cash requirements to fund other areas of the business, potentially in conjunction with less favorable funding conditions (tighter margin constraints). The uncertainty is resolved between the periods. We model the shock as an early equity withdrawal; specifically, the amount withdrawn is a Bernoulli random variable Δ such that

$$\Delta = \begin{cases} \delta & \text{with probability } \pi, \\ 0 & \text{with probability } 1-\pi. \end{cases} \tag{5}$$

Note that this additive shock model of an early withdrawal is equivalent to an increased need for cash. An investor who experiences a shock may also face tighter funding restrictions. To accommodate this possibility, we let the second period leverage limit depend on Δ and use the shorthand notation $\tilde{\rho}_2$ to denote this dependence. In this model, if the shock does not occur ($\Delta = 0$), then $\tilde{\rho}_2 = \rho_1$, i.e., the leverage limit remains as before. Otherwise, if a shock does occur ($\Delta > 0$), then $\tilde{\rho}_2 = \rho_2$, where $0 < \rho_2 \le \rho_1$, i.e., funding conditions potentially worsen.

If there is a second-period shock, liabilities increase by δ, resulting in a more leveraged balance sheet, and leverage limits, in response, potentially tighten. The investor is then required to liquidate assets in the second trading period to deleverage to within the new, allowed limits. Although more general shock distributions can be handled in the framework of the optimization problem, our goal is to obtain structural insights on the interaction between the magnitude of future distress and optimal liquidation strategies. To this end, we present the analysis of a Bernoulli shock, varying the shock magnitude δ.

Before moving forward with full details on the optimization problem, we discuss some technical assumptions on the underlying problem parameters. In everything that follows, we will assume the following.

Assumptions. *The following conditions hold*:

1. *The price impact matrices satisfy*

$$\Lambda \succ \max\left(\frac{1+\sqrt{\pi}}{2}, \frac{\rho_2 - 1}{\rho_2 + 1}\right)\Gamma, \tag{6}$$

$$((1+\rho_2)\Lambda + \Gamma)x_0 \le p_0. \tag{7}$$

2. *The trade $y_1 = -x_0/2$ generates sufficient cash to meet the first-period margin constraint*.

Condition (6) is the analog of the condition $\Lambda - \Gamma/2 \succ 0$ from the myopic problem and is only slightly more restrictive. From an optimization standpoint, it ensures that the objective function is concave in the decision variables (this is justified in the appendix). Economically, it simply means that investors cannot arbitrarily improve wealth simply through the act of trading and, as such, seems to be a reasonable requirement.[7]

Condition (7) is an upper bound on the value of the temporary price impact associated with a full fire-sale of the holdings of each asset. We are interested in analyzing shocks that may cause the investor to fully unwind holdings of certain assets. If this condition does not hold, the costs of fully liquidating a particular asset are sufficiently high that the investor would never want to do so.

Finally, we assume that a liquidation of one-half of the portfolio holdings generates sufficient cash to

[7] Note, however, that the two-period problem is guaranteed to be bounded using only the weaker assumption that we introduced for the single-period problem, $\Lambda \succ (\rho_1 - 1)/(2(\rho_1 + 1))\Gamma$. This assumption ensures strict convexity of the first-period leverage constraint, and therefore boundedness of y_1. With y_1 bounded by the first-period constraint on leverage, we only need convexity in y_2 of the second-period constraint on leverage to ensure boundedness of y_2. This is ensured by $\Lambda \succ (\rho_2 - 1)/(2(\rho_2 + 1))\Gamma$, which follows by $\Lambda \succ (\rho_1 - 1)/(2(\rho_1 + 1))\Gamma$ and $\rho_2 \le \rho_1$. This assumption is therefore sufficient to ensure that the optimal trades are finite. The same caveats as in the single-period case apply for this weaker constraint.

meet the first-period margin requirement. Given that such a trade is extreme, this assumption is weak. If this is not true, then the investor is so constrained by the initial fire sale that some of the trade-offs we discuss shortly for this two-period model may not occur.

Following the notation of the previous section, the equilibrium price after the second period is

$$\begin{aligned} p_2 &= p_1 + \Gamma y_2 \\ &= p_0 + \Gamma(y_1 + y_2), \end{aligned}$$

and the assets at the end of the second period are

$$\begin{aligned} a_2 &= p_2' x_2 \\ &= (p_0 + \Gamma(y_1+y_2))'(x_0+y_1+y_2) \\ &= a_0 + \begin{bmatrix} p_0+\Gamma x_0 \\ p_0+\Gamma x_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} \Gamma & \Gamma \\ \Gamma & \Gamma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \end{aligned}$$

Using the same price model as for the single-period problem, the investor trades y_2 for an average price of $p_1 + \Lambda y_2 + \frac{1}{2}\Gamma y_2$. After withdrawals Δ, the investor is then left with liabilities

$$\begin{aligned} l_2 &= l_1 + \Delta + (p_1 + \Lambda y_2 + \tfrac{1}{2}\Gamma y_2)' y_2 \\ &= l_0 + \Delta + \begin{bmatrix} p_0 \\ p_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} \Lambda + \frac{1}{2}\Gamma & \frac{1}{2}\Gamma \\ \frac{1}{2}\Gamma & \Lambda + \frac{1}{2}\Gamma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \end{aligned}$$

and the second-period equity, $e_2 = a_2 - l_2$, is

$$\begin{aligned} e_2 &= e_0 - \Delta + \begin{bmatrix} \Gamma x_0 \\ \Gamma x_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &\quad - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} \Lambda - \frac{1}{2}\Gamma & -\frac{1}{2}\Gamma \\ -\frac{1}{2}\Gamma & \Lambda - \frac{1}{2}\Gamma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \end{aligned}$$

The second-period leverage constraint

$$\frac{l_2}{e_2} \le \tilde{\rho}_2$$

can be written as a quadratic constraint on the vector of first- and second-period trades,

$$\begin{aligned} &\tilde{\rho}_2 e_0 - l_0 - (1+\tilde{\rho}_2)\Delta + \begin{bmatrix} \tilde{\rho}_2 \Gamma x_0 - p_0 \\ \tilde{\rho}_2 \Gamma x_0 - p_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \\ &\quad \cdot \begin{bmatrix} \tilde{\rho}_2(\Lambda - \frac{1}{2}\Gamma) + \Lambda + \frac{1}{2}\Gamma & \frac{1}{2}(1-\tilde{\rho}_2)\Gamma \\ \frac{1}{2}(1-\tilde{\rho}_2)\Gamma & \tilde{\rho}_2(\Lambda - \frac{1}{2}\Gamma) + \Lambda + \frac{1}{2}\Gamma \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \ge 0. \end{aligned} \tag{8}$$

To avoid the possibility of bankruptcy, the investor must choose their first-period trades in such a manner that the second-period constraint on leverage can be met under any realization of Δ. The two-period problem of the expected-equity maximizing investor is then

$$\begin{aligned} &\text{maximize} \quad \mathbf{E}_\Delta e_2 \\ &\text{subject to} \quad l_1 \le \rho_1 e_1, \quad l_2 \le \tilde{\rho}_2 e_2, \quad \forall\, \Delta \\ &\qquad\qquad -x_0 \le y_1 \le 0, \quad -x_1 \le y_2 \le 0, \quad \forall\, \Delta, \end{aligned}$$

where the optimization is over y_1 and y_2, where y_1 is in $\mathbb{R}^n$ and y_2 is $\{0, \delta\} \mapsto \mathbb{R}^n$ (or, equivalently, a random variable in $\mathbb{R}^n$ measurable in the sigma-algebra generated by Δ).

The problem simplifies because, when $\Delta = 0$, the optimal second-period trade is $y_2 = 0$. This can be seen by computing the gradient of the objective with respect to y_2 at $y_2 = 0$, which is Γx_1. Under assumptions of no shorting and $\gamma_i \ge 0$, all entries of the gradient are nonnegative. This, together with the concavity of the objective in y_2 (guaranteed by $\Lambda \succ \frac{1}{2}\Gamma$) and the convexity of the box constraints, ensures that, if the constraint on leverage is met, the optimum is achieved at $y_2 = 0$. The first-period leverage constraint and first-period trades in turn ensure that, absent a second-period shock, the second-period leverage constraint, which remains at a limit of ρ_1, is still satisfied. Intuitively, if there is no shock, the portfolio still meets the margin obligation and, because selling assets is costly, there is no need to further rebalance.

We therefore use $y_2 \in \mathbb{R}^n$ to denote the second-period trades associated with the realization $\Delta = \delta$. Likewise, we refer to l_2 and e_2 as the liabilities and equity when $\Delta = \delta$. Noting that when $\Delta = 0$, the optimal second-period equity is $e_2 = e_1$, we can now write the investor's expected-equity-maximization problem as

$$\begin{aligned} &\text{maximize} \quad (1-\pi)e_1 + \pi e_2 \\ &\text{subject to} \quad l_1 \le \rho_1 e_1, \quad l_2 \le \rho_2 e_2 \\ &\qquad\qquad -x_0 \le y_1 \le 0, \quad -x_1 \le y_2 \le 0. \end{aligned} \tag{9}$$

The program variables are $y_1 \in \mathbb{R}^n$ and $y_2 \in \mathbb{R}^n$, and the objective is a quadratic function in R^{2n},

$$\begin{aligned} \mathbf{E}_\Delta e_2 &= (1-\pi)e_1 + \pi e_2 \\ &= e_0 - \pi\delta + \begin{bmatrix} \Gamma x_0 \\ \pi \Gamma x_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &\quad - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} \Lambda - \frac{1}{2}\Gamma & -\pi\frac{1}{2}\Gamma \\ -\pi\frac{1}{2}\Gamma & \pi(\Lambda - \frac{1}{2}\Gamma) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \end{aligned}$$

To understand the nature of the optimal solution to this problem, it is helpful to consider the problem with the two leverage constraints dualized:

$$\begin{aligned}\text{maximize}\ \ &(1-\pi)e_1+\pi e_2+z_1(\rho_1 e_1-l_1)\\ &\quad +z_2(\rho_2 e_2-l_2) \qquad (10)\\ \text{subject to}\ \ &-x_0\le y_1\le 0,\quad -x_1\le y_2\le 0.\end{aligned}$$

If we fix the values of z_1 and z_2, the problem can be decoupled in the assets in that the optimal solution can be obtained by the independent maximization of $(y_{1,i}, y_{2,i})$ for each asset i. One can then independently derive the solution for each asset i as a function of the first- and second-period shadow prices. The objective can be written as a sum of terms associated with each asset,

$$\mathbf{E}_\Delta\, e_2 = c+\sum_{i=1}^{n}(b_i' y_i - y_i' A_i y_i),$$

where $y_i=\begin{bmatrix} y_{1,i}\\ y_{2,i}\end{bmatrix}$. The constant term c and the linear and quadratic terms b_i and A_i depend on the first- and second-period shadow prices z_1 and z_2, and are as follows,

$$\begin{aligned} c&=e_0-\pi\delta+(\rho_1 e_0-l_0)z_1+(\rho_2 e_0-l_0)z_2-(1+\rho_2)\delta z_2,\\ b_i&=\begin{bmatrix}(1+\rho_1\gamma_i z_1+\rho_2\gamma_i z_2)x_{0,i}-(z_1+z_2)p_{0,i}\\ (\pi+\rho_2\gamma_i z_2)x_{0,i}-z_2 p_{0,i}\end{bmatrix},\\ A_{i,11}&=(\lambda_i-\tfrac{1}{2}\gamma_i)(1+\rho_1 z_1+\rho_2 z_2)+(\lambda_i+\tfrac{1}{2}\gamma_i)(z_1+z_2),\\ A_{i,12}&=-\tfrac{1}{2}\pi\gamma_i-\tfrac{1}{2}(\rho_2-1)\gamma_i z_2,\\ A_{i,21}&=-\tfrac{1}{2}\pi\gamma_i-\tfrac{1}{2}(\rho_2-1)\gamma_i z_2,\\ A_{i,22}&=(\lambda_i-\tfrac{1}{2}\gamma_i)(\pi+\rho_2 z_2)+(\lambda_i+\tfrac{1}{2}\gamma_i)z_2.\end{aligned}$$

Because $x_1 = x_0 + y_1$, the constraints disallowing position increases and short sales can be equivalently stated as

$$y_{1,i}\le 0,\quad y_{2,i}\le 0,\quad \text{and}\quad y_{1,i}+y_{2,i}\ge -x_{0,i}.$$

This defines a triangular feasible set for the first- and second-period trades in each asset. In short, conditional on the optimal values of the shadow prices z_1 and z_2 of the margin constraints in both periods, the problem decouples into n optimization problems (one per asset), with each problem maximization of a concave quadratic objective over three linear constraints. We derive the solution in detail in the appendix.

3.2. Preemptive Deleveraging and Optimal Liquidation

We now proceed to characterize optimal trading in this setting that allows for a downstream shock. The first important question that we address is whether the leverage constraint

$$\frac{l_1}{e_1}\le\rho_1$$

binds in the first period. That is, we consider whether the investor deleverages preemptively (i.e., more than is immediately required) in the first period when there is a potential need for liquidity in the future. The following result addresses this question.

Result 3. *Suppose that the investor's initial holdings are such that $l_0/e_0>\rho_1$, i.e., immediate deleveraging is required. Then there exists a threshold shock level $\hat{\delta}\ge 0$ such that the optimal two-period solution satisfies $l_1/e_1=\rho_1$ for all $\delta\in[0,\hat{\delta}]$ and $l_1/e_1<\rho_1$ for all feasible $\delta>\hat{\delta}$.*

Result 3 says that the optimal two-period liquidator may in fact preemptively deleverage beyond what is required in the first period. When the potential need for future liquidity is large, the margin constraint does not bind in the first period. Because of the convexity of the penalty incurred in a large fire sale, the risk-neutral investor manages future liquidity risk by overliquidating the portfolio early on. In Example 2 below, we find that this effect becomes more pronounced for a risk-averse investor. Recall that in the case of myopic deleveraging as in §2.2, such preemptive deleveraging does not occur.

Another interpretation of Result 3 is that, if the expected future need for liquidity is high enough, the investor substitutes liquid assets for illiquid ones. If we interpret cash to be the $(n+1)$th asset in the portfolio, the investor overweights this (most liquid) asset to the detriment of other securities.

Given that Result 3 deals with the possibility of leaving a margin buffer as protection against a possible shock, it is interesting to place this in the context of risk constraints (e.g., VaR), which are often geared toward computing appropriate capital buffers.[8] For instance, under the convention that $\text{VaR}_\alpha(X)$ is the minimum amount of capital to be added to a position X so that the position breaks even with probability at least $1-\alpha$, one can view the preemptive deleveraging in Result 3 in terms of a VaR constraint on the second-period liabilities. To see this connection more explicitly, consider the constraint

$$\text{VaR}_\pi(-l_2)\le 0,$$

where we flip the sign of liabilities (because they are losses). Note that $l_2=l_1+\Delta-D(y_1,y_2)$, where $D(y_1,y_2)$ is the liability decrease from second-period

[8] We thank an anonymous referee for making this connection, which led us to look more closely at the relationship of our work to standard VaR constraints.

trade (which depends on y_1 as well through p_1). Because a nonnegative shock δ occurs with probability π, we have

$$\begin{aligned}\mathrm{VaR}_\pi(-l_2) &= \mathrm{VaR}_\pi(-l_1 - \Delta + D(y_1, y_2))\\ &= l_1 + \delta - D(y_1, y_2),\end{aligned}$$

so that

$$\mathrm{VaR}_\pi(-l_2) \le 0 \iff D(y_1, y_2) \ge l_1 + \delta,$$

i.e., the VaR constraint is equivalent to requiring that l_1 be reduced enough such that it is feasible, through second-period deleveraging, to cover the potential shock. If δ is fairly small, this is a cheap requirement to meet, and the investor can set l_1 as high as possible (i.e., $l_1 = \rho_1 e_1$). Otherwise, when δ is large, it becomes too costly (or infeasible) to deleverage by that extent in the second period, and the investor must introduce some slack into the constraint by reducing l_1 (i.e., $l_1 < \rho_1 e_1$). Note also that models that understate the cost of trading (by, say, ignoring price impact altogether, or assuming a longer trading horizon), effectively overstate $D(y_1, y_2)$ here and therefore may lead to unrealistically low risk assessments.

Preemptive deleveraging like this is, in some sense, to be intuitively expected when shocks are large. In a portfolio setting, perhaps more interesting is the question of how the investor should prioritize assets for liquidation given the possibility of a downstream shock. The first finding we report is that, unlike the case of myopic deleveraging, it is no longer the case that the investor simply sells more of the liquid assets. Moreover, all else equal, the temporary and permanent components of price impact have, in some sense, opposing effects on the optimal solution.

Result 4. *Let i and j be any two assets with equal initial prices and equal initial holdings. Then for any $\pi \in (0, 1)$, $0 < \rho_2 \le \rho_1$, the following hold:*

(a) *If $\lambda_i = \lambda_j$ and $\gamma_i < \gamma_j$, then, for any strictly feasible $\delta > 0$, lower permanent price impact assets are prioritized for liquidation, i.e., $y^*_{1,i} < y^*_{1,j}$.*

(b) *If $\gamma_i = \gamma_j$ and $\lambda_i > \lambda_j$, then for sufficiently large δ, higher temporary price impact assets are prioritized for liquidation, i.e., $y^*_{1,i} < y^*_{1,j}$.*

The first statement says that, all else equal, the investor will always want to trade more of assets that have a low permanent price impact, no matter how great the need for future liquidity may be. However, the second statement says that this does not hold for the temporary component of liquidity: if the future shock is sufficiently large, then the investor optimally retains more of assets that have a small temporary component of liquidity in the first period in preparation for the possibility of future distress.

An intuition for these opposing effects can be gained as follows. If the shock does occur, the decrease in final equity due to the permanent impact depends only on the total amount sold over the two periods, and not on the way this total amount is traded over time. When assets differ only in their permanent price impacts, there is no benefit from initially selling more of the high permanent impact assets, as this will hurt equity if the shock does not occur.

For temporary price impact, however, the decrease in final equity does depend on the way the assets are traded over time. This component of liquidity instead encourages the investor to smooth trades as much as possible over the two periods. Because of this, when the potential shock is large enough, it is optimal for the investor to obtain a larger portion of their first-period deleveraging from assets with high temporary price impact, all else equal.

In short, optimal liquidation may involve retaining assets with low temporary price impact as a hedge against a crisis. This is not the case for assets with low permanent impact: the investor prefers to sell more of such assets. Such liquidation behavior is a clear departure from the case of myopic deleveraging.

Result 4 looks at two cases in which one of the components of price impact is constant across assets. As discussed in the introduction, these components of price impact arise from different sources, and empirical evidence (e.g., Sadka 2006) points to considerable variation across these two components. Given this, it is of interest to look at the structure of optimal liquidation trades when both components vary across assets. Result 4 is a special case of the following more general result that describes the interplay between permanent and temporary price impact when liquidating under risk of a recurrent shock.

Result 5. *Let i and j be any two assets with equal initial prices and equal initial holdings. Then for any $\pi \in (0, 1)$, $0 < \rho_2 \le \rho_1$, the following hold:*

(a) *If asset i is more liquid than asset j ($\lambda_i < \lambda_j$, $\gamma_i < \gamma_j$) and, in addition, $\gamma_i/\lambda_i < \gamma_j/\lambda_j$, then, for any strictly feasible $\delta > 0$, asset i is prioritized for liquidation, i.e., $y^*_{1,i} < y^*_{1,j}$.*

(b) *If $\gamma_i/\lambda_i > \gamma_j/\lambda_j$, then for sufficiently large δ, asset j is prioritized for liquidation, i.e., $y^*_{1,j} < y^*_{1,i}$.*

Part (a) of Result 5 aligns with what we have found for the optimal liquidation behavior described for the case of myopic deleveraging (§2.2). Namely, even accounting for the possibility of future distress, it is optimal for the investor to sell more of assets that are more liquid, no matter how extreme the future shock may be. For this result to hold, however, note that an extra condition is required: the more liquid asset needs to have a lower ratio of permanent to temporary price impacts.

Part (b) of Result 5 shows that this ratio condition is not merely sufficient, but it is also *necessary* to ensure prioritization of one asset over another in the optimal trades when facing the prospect of a large shock. For a large enough future shock, the ratio of permanent to temporary price impact drives the investor's optimal trades more than the price impacts considered individually. In this case, the investor may favor selling less liquid assets in the first period to hedge against the future need for liquidity.

It is not obvious that a strategy that immediately sells more of illiquid assets could ever be optimal. Intuition may suggest that the investor, as a hedge against future distress, would increase the *relative* proportion of illiquid to liquid assets sold in the first period, yet still sell a greater absolute amount of the more liquid assets. This may reasonably be expected, as the future shock is not sure to occur, whereas first-period trades result in immediate costs that are certain. Yet this alone is not necessarily optimal: for sufficiently large shocks, an optimal liquidator chooses to unload a strictly larger quantity of assets with a low ratio of permanent to temporary price impact, even if these assets are less liquid across both components of price impact.

It is also somewhat surprising that a fairly simple index—the ratio of permanent to temporary price impact—emerges from the analysis characterizing the structure of optimal liquidation behavior. Some intuition for why this ratio is crucial can be gained by extending the reasoning in the discussion following Result 4. If the shock does not occur, the problem is essentially like the one faced by the myopic liquidator: in this case, the investor wants to sell as little as possible to keep trading costs low and to avoid depressing, as much as possible, the price of assets currently held. On the other hand, if a shock does occur, deleveraging may be expensive, primarily due to the dominant effect of temporary price impact. In this case, the investor would prefer to smooth trades as much as possible over the two periods to avoid large and costly downstream rebalancing.

The ratio of permanent to temporary impacts represents, in some sense, a normalized measure of efficiency in balancing between these two competing effects: Namely, if the ratio of permanent to temporary price impacts is low, the investor can smooth trades to hedge against future distress in a manner that is relatively cheap in comparison to the resulting depression of prices. When the future shock is sufficiently large, even if it is unlikely, the marginal cost of future distress becomes prohibitive enough to tilt the investor toward favoring sales of assets that are, in this sense, efficient.

The fact that the ratio of price impacts drives the liquidation strategy more generally underscores the finding from Result 4: If the future shock is potentially severe, permanent and temporary price impact influence optimal liquidation strategies in opposite ways. This ratio arises as the relevant index here due to the quadratic structure of the liquidation problem, which in turn depends on some of the specific assumptions in our model (such as a linear price impact function and risk neutrality). Although such a simple index may not arise in a more general model, we would expect the opposing influence of permanent and temporary impact for large shocks to be qualitatively similar in such settings.

To demonstrate our findings more concretely, we now revisit the illustrative example from earlier.

Example 2. Consider the same parameters as in Example 1. Recall that the assets are in order of increasing liquidity (from 1 to 4) in both temporary and permanent impact. On the other hand, we have

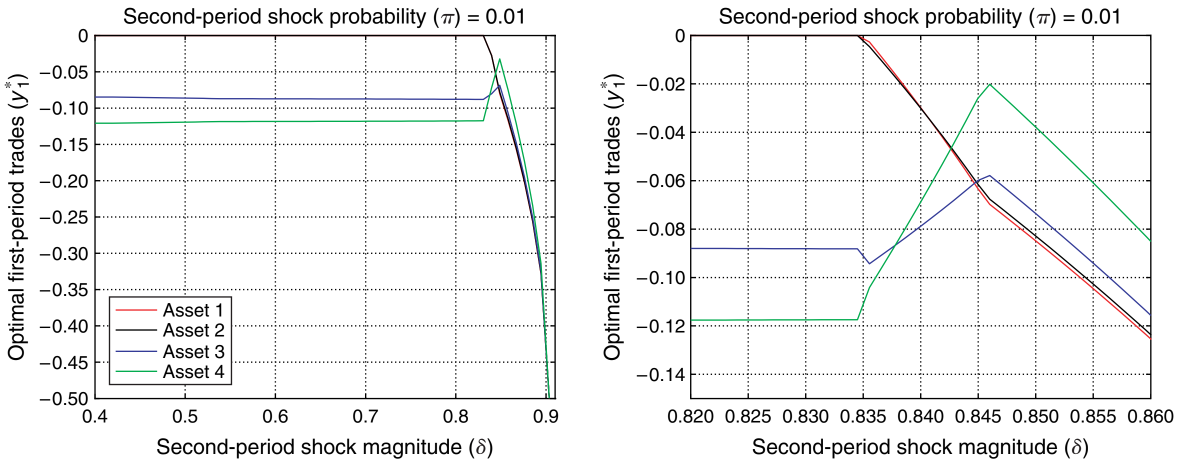
$$\frac{\gamma_1}{\lambda_1} < \frac{\gamma_2}{\lambda_2} < \frac{\gamma_3}{\lambda_3} < \frac{\gamma_4}{\lambda_4}.$$

Because of this ordering of the ratios, part (b) of Result 5 shows that, no matter how likely future distress may be, there is always a sufficiently large shock such that a larger amount of asset i will be liquidated than asset $j > i$, even though asset j is more liquid than asset i. Recall that the optimal myopic solution is $y^*_{1,1} = 0$, $y^*_{1,2} = 0$, $y^*_{1,3} = -0.0851$ M, and $y^*_{1,4} = -0.1205$ M.

Figure 1 shows the optimal first-period trades y^*_1 versus the second-period shock magnitude (δ) for the case when the shock is believed to be fairly unlikely ($\pi = 0.01$). One can see that if the possible shock is large enough, the optimal solution involves selling more of the relatively illiquid assets—in the figure detail on the right, it is apparent that for $\delta > \$0.85$ M, less of the most liquid asset (asset 4) is sold, and, in fact, the ordering of assets by amount sold is the inverse of their ordering by liquidity. Note that in this example, we are (conservatively) setting $\rho_2 = \rho_1 = 18$; if we set ρ_2 to smaller values, this would result in more preemptive deleveraging and higher sales of the illiquid assets for the same shock size. The shock alone is sufficient to drive these trading patterns, and one would expect such trades to be more severe if funding conditions also strictly worsen, because the investor would then trade less myopically in the first period due to the possibility of increased restrictions.

Also note from Figure 1 that the amounts of each asset sold need not be nondecreasing in δ (though the total trade size, summing across all assets, will be). In particular, we see that around $\delta = \$0.83$ M, the optimal liquidation strategy involves selling *less* of the most liquid asset as δ increases: over this range, it actually becomes optimal to hedge the post-trade position by selling more of the relatively illiquid assets and retaining a larger stake in the liquid asset.

Figure 1 **Optimal Trades for Example 2 with Future Distress Relatively Unlikely ($\pi = 0.01$) (Detail on the Right)**



It is notable that the optimal trades may be so different than the myopic trades, and these differences have significant implications in terms of future ruin. For instance, for $\delta = \$0.85$ M, the optimal first-period trade is $y^*_{1,1} = -0.0849$ M, $y^*_{1,2} = -0.0829$ M, $y^*_{1,3} = -0.0736$ M, and $y^*_{1,4} = -0.0379$ M. The net equity after the first period is now $e^*_1 = \$0.9860$ M, which is \$0.0125 M worse than the the first period equity following the optimal myopic strategy (or 1.25% of the original net equity). However, in the event of a shock, an investor following the myopic strategy goes bankrupt. This is not the case for the anticipative liquidator, who adjusts positions in a manner that incurs slightly higher immediate costs but retains a more liquid portfolio.

Figure 2 shows a similar illustration in the case where future distress is considerably more likely ($\pi = 0.25$). Most of the phenomena discussed above are still present in this case but are even more pronounced. For instance, the crossover δ when more of less liquid asset is sold relative to the most liquid asset is now much smaller. Moreover, because a shock is considerably more likely, the investor is now much more willing to sell off amounts of the two least liquid assets (assets 1 and 2). Comparing the results for the cases $\pi = 0.01$ and $\pi = 0.25$, we see that the investor's assessment of the likelihood of future distress greatly impacts the structure of the trades, which underscores the fact that the modeling of the probability of continuation or recurrence of a shock plays a crucial role in optimal liquidation.

Figure 3 shows the leverage ratio of the optimal liquidator's position following first-period trade as δ grows, which illustrates Result 3. For large enough δ, it is optimal for the investor to trade more than required by the margin constraint. Not surprisingly, this happens for a smaller δ when the shock is more likely, because the investor will be more cautious as π

Figure 2 **Optimal Trades for Example 2 with Future Distress Probability $\pi = 0.25$**

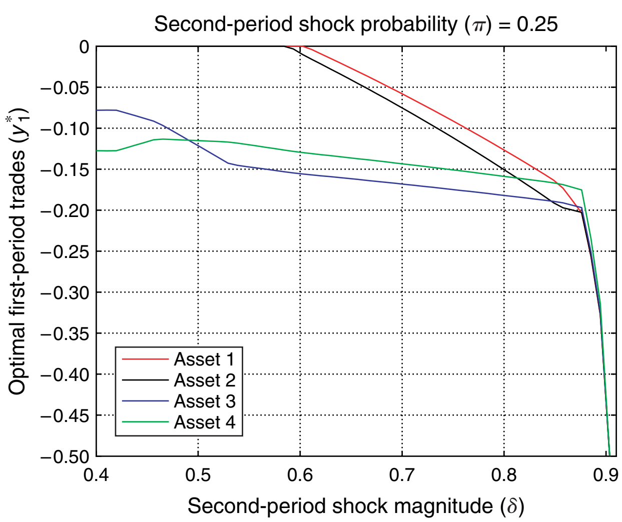


Figure 3 **First-Period Post-Trade Leverage Ratios for Example 2**

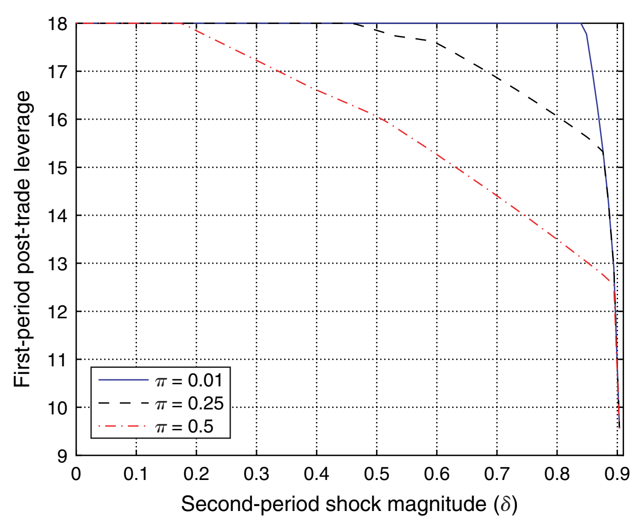
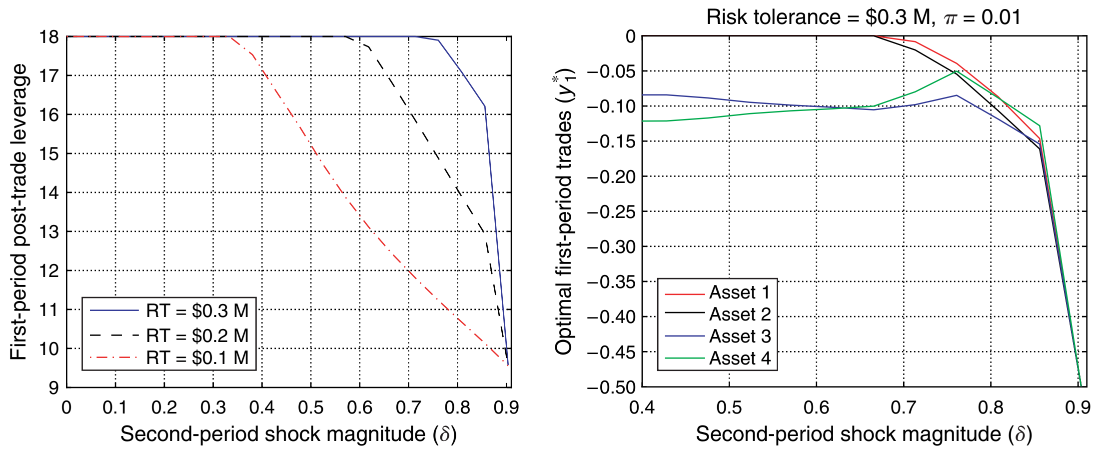


Figure 4 **Results for Example 2 with CARA Utility and $\pi = 0.01$**



grows. Note that the kink in sales of the liquid asset in Figure 1 near $\delta = \$0.85$ M corresponds to the point where the first-period margin constraint is no longer binding.

Finally, we consider an example of how risk aversion may make these effects more pronounced. Figure 4 shows results for $\pi = 0.01$ in the case of a risk-averse liquidator. Specifically, we examine the case of an investor with an exponential utility function (i.e., constant absolute risk aversion (CARA)). We would expect the amount of deleveraging to increase with the investor's risk aversion, and the smallest shock size that leads to preemptive deleveraging to decrease. The plot on the left is consistent with this intuition; as the investor becomes more risk averse, more preemptive deleveraging results. The plot on the right shows the optimal first-period trades in the case of a risk tolerance level of \$0.3 M. Compared to the risk-neutral case, optimal liquidation involves selling more of less liquid assets for smaller potential shocks. The investor optimally liquidates more of asset 3 than of asset 4 when the potential shock is larger than $\delta \approx \$0.6$ M, and begins to sell some of the least liquid assets (assets 1 and 2) when the potential shock is larger than $\delta \approx \$0.68$ M.

4. Conclusion

Hedging future distress risk is crucial for portfolio management, especially during financial crises. This requires an understanding of the trade-offs between, on the one hand, the immediate cost of selling illiquid assets and, on the other, the risk of retaining an illiquid portfolio. We have studied how the liquidity characteristics of the assets in a portfolio affect optimal liquidation in the presence of financial pressures by analyzing a two-period problem in which an investor faces present and possibly future liquidity needs. The assets in the investor's portfolio differ along two liquidity dimensions: their temporary and permanent price impacts of trading. The investor's goal is to maximize portfolio value while meeting present liquidity constraints and planning for downstream shocks. We find that, when the risk of future distress is high, it is optimal to sell more of assets with a lower ratio of permanent to temporary price impact.

Whether large investors take such considerations into account during crisis remains an open empirical question. Manconi et al. (2010) study the behavior of institutional investors holding AAA-rated securitized bonds and lower-rated corporate bonds during the financial crisis of 2007–2008. They find that, at the onset of the crisis, more constrained investors retained the (now illiquid) asset-backed securities, believed to have higher permanent price impact, and sold off low-rated corporate bonds, believed to have higher temporary price impact. Although this is consistent with our analysis, more empirical is needed, especially in markets for which it is possible to partition securities according to their temporary and permanent price impact characteristics using the approach in Sadka (2006).

Finally, as noted in the introduction, our analysis implies that investors may benefit from including the cost of forced deleveraging of a large portfolio as a consideration in portfolio selection, as well as from maintaining contingency plans for its partial unwinding. Including such considerations in computing measures of risk may be worthwhile, especially when analyzing model formulations that include more complex uncertainty and multiple time periods.

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Appendix. Proofs

Proof of Result 1

Consider the problem with the leverage constraint dualized,

$$\begin{aligned} &\text{maximize} \quad e_1 + z(l_1 - \rho_1 e_1) \\ &\text{subject to} \quad -x_0 \le y_1 \le 0. \end{aligned}$$

With diagonal structure the assets are decoupled, in that the solution can be obtained by separately optimizing the trades in each asset. If the box constraint in asset i is not active, its solution is obtained from the first-order condition that, from problem convexity, is sufficient. Increasing ρ_1 (or decreasing l_0) relaxes the margin constraint, which implies a smaller shadow price z, from which the monotonicities follow by simple algebra. (The nonmonotonicities can be verified by simple counter examples.) □

Proof of Result 2

We show the results by arguing that the partial derivative of $y^*_{1,i}$ as λ_i, γ_i, or $x_{0,i}$ varies in the appropriate direction, noting that the shadow price z will be constant, as we are analyzing a fixed trade. The claims for λ_i and $x_{0,i}$ are immediate from the expression for $y^*_{1,i}$. Consider now γ_i. Because the expression only involves asset i (the constancy of z ensures that the assets remain decoupled), we can without loss of generality normalize $p_{0,i} = 1$ and $x_{0,i} = 1$. The numerator of the partial derivative $\partial y^*_{1,i}/\partial \gamma_i$ can then be seen to simplify to

$$2\lambda_i + (2\lambda_i - 1)z + z^2.$$

Therefore, the partial derivative is positive for any $z < z_0$ where

$$z_0 = \tfrac{1}{2}\left(1 - 2\lambda_i - \sqrt{4\lambda_i^2 - 12\lambda_i + 1}\right),$$

if $\lambda_i \le \frac{3}{2} - \sqrt{2}$, and for all z otherwise. From the constraint on short sales $y^*_{1,i} \ge -1$, the dual variable z must satisfy

$$z \le \frac{2\lambda_i}{1 - 2\lambda_i - \gamma_i} \le \frac{2\lambda_i}{1 - 4\lambda_i}.$$

The proof is completed by showing that this is less than z_0 for any $\lambda_i > 0$. The inequality can be written as

$$8\lambda_i^2 - 10\lambda_i + 1 > (1 - 4\lambda_i)\sqrt{4\lambda_i^2 - 12\lambda_i + 1}.$$

From inspection of the roots of the different factors, both sides are positive for $\lambda_i \le \frac{3}{2} - \sqrt{2}$, and we can therefore square both sides. Collecting terms, the inequality then simplifies to $64\lambda_i^3 > 0$.

The two claims in the result are then immediate. □

Proof of Condition (6)

Consider two matrices A and B in $\mathbb{R}^{n\times n}$ such that $A = A'$, $B = B'$, $A \succ 0$, and $B \succ 0$. We show that the matrix

$$M = \begin{bmatrix} A & -B \\ -B & A \end{bmatrix}$$

is positive definite if and only if $A \succ B$. Because $A \succ 0$, M is positive definite if and only if its Schur complement is positive definite: $A - BA^{-1}B \succ 0$. Because $A \succ 0$, by change of coordinates this is equivalent to $I - (A^{-1/2}BA^{-1/2})^2 \succ 0$. Because $A^{-1/2}BA^{-1/2}$ is symmetric and therefore has identical left and right eigenvectors, the condition is true if and only if all the eigenvalues of $A^{-1/2}BA^{-1/2}$ satisfy $\lambda^2 < 1$. Because $A^{-1/2}BA^{-1/2}$ is positive definite, this is equivalent to $\lambda < 1$. We conclude that we can write the condition on the Schur complement as $I - A^{-1/2}BA^{-1/2} \succ 0$. By change of coordinates, this is equivalent to $A - B \succ 0$.

Applying this result to the matrix in the quadratic form in (8) leads to the condition

$$\Lambda \succ \frac{\rho_2 - 1}{\rho_2 + 1}\Gamma.$$

By change of coordinates, the matrix in the quadratic form in $\mathbf{E}_\Delta e_2$ is positive definite if

$$\begin{bmatrix} \Lambda - \frac{1}{2}\Gamma & -\sqrt{\pi}\frac{1}{2}\Gamma \\ -\sqrt{\pi}\frac{1}{2}\Gamma & \Lambda - \frac{1}{2}\Gamma \end{bmatrix}$$

is positive definite. Applying the result above to this matrix leads to the condition

$$\Lambda \succ \frac{1 + \sqrt{\pi}}{2}\Gamma. \quad \square$$

Proof of Result 3

Notice that for $\delta = 0$, the two-period problem is identical to the myopic deleveraging problem. Thus, at $\delta = 0$ it must be the case for the optimal solution to satisfy $l_1/e_1 = \rho_1$. Suppose not, i.e., suppose at $\delta = 0$, we have $l_1/e_1 < \rho_1$. Then the shadow price of the first-period margin constraint is zero, and the problem is equivalent to a problem that maximizes the net equity subject to $-x_0 \le y_1 \le 0$. Because we have $y_1 \le 0$, $\Gamma \ge 0$, and $x_0 \ge 0$, it follows that

$$\begin{aligned} e_1 &= e_0 + x_0'\Gamma y_1 - y_1(\Lambda - \tfrac{1}{2}\Gamma)y_1 \\ &\le e_0 - y_1(\Lambda - \tfrac{1}{2}\Gamma)y_1 \\ &\le e_0, \end{aligned}$$

with equality strict for any nonzero y_1 due to $\Lambda - \Gamma/2 \succ 0$. This means $y_1 = 0$ must be optimal, which is a contradiction, as the trader initially satisfied $l_0/e_0 > \rho_1$.

Now consider the maximum possible δ such that the two-period problem is still feasible. We will consider the problem of finding the maximum such δ and show that at this value, it must be that $l_1/e_1 < \rho_1$. Denote this maximum feasible δ by $\bar{\delta}$; one can see that $\bar{\delta}$ is given by

$$\bar{\delta} = \frac{\rho_2 a_0 - (\rho_2 + 1)l_0 + \rho_2 v^*}{\rho_2 + 1},$$

where v^* is the optimal value of the (convex) problem

$$\underset{y_1, y_2}{\text{maximize}} \quad -(p_0 - \rho_2\Gamma x_0)(y_1 + y_2) - [y_1' y_2'] \cdot \begin{bmatrix} (\rho_2+1)\left(\Lambda+\frac{1}{2}\Gamma\right)-\rho_2\Gamma & \frac{1-\rho_2}{2}\Gamma \\ \frac{1-\rho_2}{2}\Gamma & (\rho_2+1)\left(\Lambda+\frac{1}{2}\Gamma\right)-\rho_2\Gamma \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{aligned} \text{subject to} \quad & \rho_1 a_0 - (\rho_1+1)l_0 - (p_0 - \rho_1\Gamma x_0)' y_1 \\ & \quad - y_1'((\rho_1+1)(\Lambda + \tfrac{1}{2}\Gamma) - \rho_1\Gamma)y_1 \geq 0 \\ & -x_0 \leq y_1 \leq 0 \\ & -x_0 - y_1 \leq y_2 \leq 0. \end{aligned}$$

We will show that the optimal solution will strictly satisfy the first-period margin constraint, so we will omit this constraint for now as we consider computation of $\bar{\delta}$. Define the following for ease of notation:

$$d \triangleq p_0 - \rho_2\Gamma x_0,$$

$$D \triangleq \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \begin{bmatrix} (\rho_2+1)\left(\Lambda+\frac{1}{2}\Gamma\right)-\rho_2\Gamma & \frac{1-\rho_2}{2}\Gamma \\ \frac{1-\rho_2}{2}\Gamma & (\rho_2+1)\left(\Lambda+\frac{1}{2}\Gamma\right)-\rho_2\Gamma \end{bmatrix}.$$

Because $d \geq 0$ (by assumption; recall that for any asset i for which $d_i \leq 0$, we can remove it from the original problem without loss of generality) and $D \succ 0$ (strict convexity), it can never be optimal to have $y_1 > 0$ or $y_2 > 0$ above, and therefore we can ignore the nonpositivity constraints.

We thus focus on finding the optimal value to the problem

$$\begin{aligned} \underset{y_1, y_2}{\text{maximize}} \quad & -d'(y_1 + y_2) - [y_1' \; y_2'] D \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \text{subject to} \quad & y_1 + y_2 + x_0 \geq 0. \end{aligned}$$

Using Lagrange multipliers $\nu \geq 0$ for the inequality constraints, the Lagrangian to this problem is given by

$$\mathscr{L}(y_1, y_2, \nu) = x_0'\nu + (\nu - d)'(y_1 + y_2) - [y_1' y_2'] D \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

and the optimal solution for any ν is given by

$$\begin{bmatrix} y_1(\nu) \\ y_2(\nu) \end{bmatrix} = \tfrac{1}{2} D^{-1} \begin{bmatrix} \nu - d \\ \nu - d \end{bmatrix}.$$

If we can find a $\nu \geq 0$ such that the corresponding solution is also feasible, then it must be optimal. Let $H = A + B$ and $\nu = d - Hx_0$. We have

$$\begin{aligned} \nu &= d - (A + B)x_0 \\ &= p_0 - \rho_2\Gamma x_0 - ((\rho_2+1)\Lambda + (1-\rho_2)\Gamma)x_0 \\ &\geq p_0 - \rho_2\Gamma x_0 - (p_0 - \Gamma x_0 + (1-\rho_2)\Gamma x_0) \\ &= 0, \end{aligned}$$

where in the inequality we are using the condition that the total price impact be sufficiently small, i.e., $(\rho_2 + 1)\Lambda x_0 \leq p_0 - \Gamma x_0$. Thus, $\nu \geq 0$ for this choice. Moreover,

$$\begin{aligned} \begin{bmatrix} y_1(\nu) \\ y_2(\nu) \end{bmatrix} &= \tfrac{1}{2} D^{-1} \begin{bmatrix} \nu - d \\ \nu - d \end{bmatrix} \\ &= -\tfrac{1}{2} D^{-1} \begin{bmatrix} H^{-1}x_0 \\ H^{-1}x_0 \end{bmatrix} \\ &= -\tfrac{1}{2} \begin{bmatrix} x_0 \\ x_0 \end{bmatrix}. \end{aligned}$$

Clearly, splitting up the trade in half over the two periods satisfies the no-short sales constraints. Moreover, by assumption, it strictly satisfies the first-period margin constraint. Thus, this solution is feasible and therefore optimal to the above problem for finding v^*, and hence the maximum level $\bar{\delta}$.

Notice that strict convexity of the objective implies that this is the only optimal solution to this problem; hence, at $\delta = \bar{\delta}$, the trade $(y_1, y_2) = (-x_0/2, -x_0/2)$ is the only feasible solution. Because it satisfies the first-period margin constraint strictly, we have found a large enough δ such that the optimal solution to the two-period problem satisfies $l_1/e_1 < \rho_1$.

Thus, at $\delta = 0$ we have $l_1/e_1 = \rho_1$ and at $\delta = \bar{\delta}$ we have $l_1/e_1 < \rho_1$; the shadow price z_1 associated with the first-period margin constraint is a continuous and nonincreasing function of δ. This implies that at some $\delta \in (0, \bar{\delta})$, the shadow price goes to zero, proving the threshold property that was claimed. □

Discussion of Form of Two-Period Optimal Solution

As noted, the feasible set for the first- and second-period trades in each asset is triangular. We graph the possible cases in Figure A.1 and derive their optimal trades in Table A.1. Depending on which constraints are active and inactive, there are seven different cases to consider.

Figure A.1 Enumeration of Cases for the Linear Constraints on Each Asset

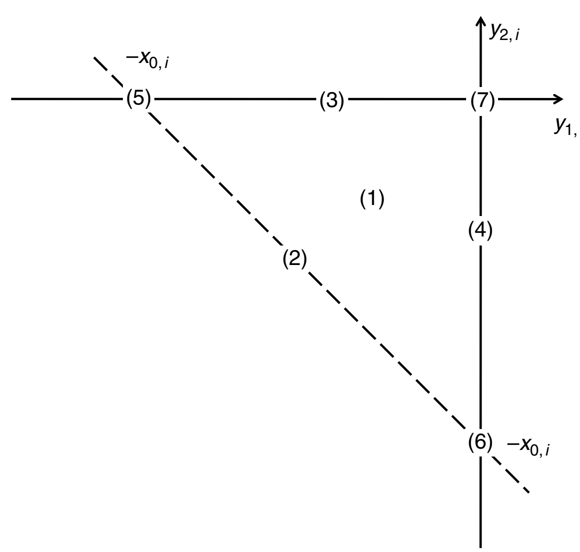

Table A.1 Solution Form for Each of the Cases in Figure A.1

Case	$y^*_{1,i}$	$y^*_{2,i}$
1	$\dfrac{A_{i,2,2}b_{i,1}-A_{i,1,2}b_{i,2}}{2(A_{i,1,1}A_{i,2,2}-A^2_{i,1,2})}$	$\dfrac{-A_{i,1,2}b_{i,1}+A_{i,1,1}b_{i,2}}{2(A_{i,1,1}A_{i,2,2}-A^2_{i,1,2})}$
2	$\dfrac{b_{i,1}-b_{i,2}+2x_{0,i}(A_{i,1,2}-A_{i,2,2})}{2(A_{i,1,1}+A_{i,2,2}-2A_{i,1,2})}$	$\dfrac{b_{i,2}-b_{i,1}+2x_{0,i}(A_{i,1,2}-A_{i,1,1})}{2(A_{i,1,1}+A_{i,2,2}-2A_{i,1,2})}$
3	$\dfrac{b_{i,1}}{2A_{i,1,1}}$	0
4	0	$\dfrac{b_{i,2}}{2A_{i,2,2}}$
5	$-x_{0,i}$	0
6	0	$-x_{0,i}$
7	0	0

Which case occurs depends on the amount of deleveraging required immediately, the size and likelihood of a potential shock in the second period, and the liquidity parameters and holdings of a particular asset.

The first two cases are the most interesting, as the remaining five correspond to more extreme situations. In Case 1, the optimal solution is strictly in the interior of the triangle. If an asset is in this region, it has favorable enough liquidity properties that is optimal to sell some of it in the first period, and some of it in the second period if a subsequent need for liquidity arises. However, even if a shock does occur in the second period, the investor will not fully liquidate the asset. Case 2 is similar to Case 1, except that if the shock occurs, the investor is forced to liquidate the entire stake in the asset. The investor would prefer to sell more of the asset, but the no-short sales constraint binds and changes the character of the optimal solution.

In Case 3, the investor chooses to liquidate some of the asset in period one but nothing further in period two if a shock occurs. This may occur, for instance, if a large amount of deleveraging is required immediately but the subsequent size of Δ is small. Case 5 is similar except that the investor liquidates all of the asset immediately. This may occur if the first-period deleveraging is large and the asset is very liquid.

In Case 4, the investor does not sell the asset in the first period, but does sell some of it in the second period. This might occur if the asset is relatively illiquid but, due to limitations on positions of the other assets, the investor has no choice but to sell some of it if a second-period shock occurs. Case 6 is similar to Case 4 in that the asset is not sold in the first period, but is completely liquidated in the second period. This might occur if Δ is very large. Finally, in Case 7, the investor does not liquidate any of the asset. This might happen if the asset is highly illiquid or if Δ is very small.

Proof of Result 4
Both parts are corollaries of Result 5. □

Proof of Result 5
Part (a): We prove the result for the interesting cases in Table A.1 (Cases 1 and 2); the proof for Cases 3–7 follows in similar fashion.

We will show that $y^*_{1,i} < y^*_{1,j}$ holds for any such δ and π in the three situations below, which cover all possibilities at the optimal solution:

1. The no-short sales constraints are not binding for either asset i or asset j (i.e., both are in Case 1).

2. The no-short sales constraint is binding for asset i but not asset j (asset i is in Case 2 and asset j is in Case 1; we will argue the intuitive fact that the reverse situation cannot occur).

3. The no-short sales constraints are binding for both assets i and j (both assets are in Case 2).

Situation 1. Assume that at the optimal solution, the no-short sales constraints are not binding for either asset i or asset j, and assume that $y^*_{1,i} \geq y^*_{1,j}$. We will show that such a solution cannot be optimal. To this end, we will show there exists a feasible direction from (y^*_1, y^*_2) in which we can head and strictly improve the objective. This implies that the solution cannot be optimal.

In particular, for some $\epsilon > 0$, let ϵ_1 be the vector with zeros everywhere but $-\epsilon$ in entry i and $+\epsilon$ in entry j. Notice that, for a sufficiently small ϵ, the solution $(y^*_1+\epsilon_1, y^*_2)$ still satisfies the no-short sales constraints.

Now we examine the gradients of the liability amount and the net equity in each period. For the first period, we have

$$\nabla l_1(y_1) = p_0 + (2\Lambda+\Gamma)y_1,$$
$$\nabla e_1(y_1) = \Gamma x_0 - (2\Lambda-\Gamma)y_1,$$

and thus, because we have $0 \geq y^*_{1,i} \geq y^*_{1,j}$, $\lambda_i < \lambda_j$, and $\gamma_i < \gamma_j$,

$$\epsilon_1' \nabla l_1(y^*_1)/\epsilon = -(2\lambda_i+\gamma_i)y^*_{1,i} + (2\lambda_j+\gamma_j)y^*_{1,j} \leq 0,$$

and (recalling that convexity requires $2\lambda_i - \gamma_i \geq 0$ for all i):

$$\begin{aligned}\epsilon_1' \nabla e_1(y^*_1)/\epsilon &= (\gamma_j-\gamma_i)x_0 - (2\lambda_j-\gamma_j)y^*_{1,j} + (2\lambda_i-\gamma_i)y^*_{1,i}\\ &\geq (\gamma_j-\gamma_i)x_0 - (2(\lambda_j-\lambda_i)-(\gamma_j-\gamma_i))y^*_{1,j}\\ &\geq \begin{cases}(\gamma_j-\gamma_i)x_0 & \text{if } 2(\lambda_j-\lambda_i)-(\gamma_j-\gamma_i)\geq 0,\\ 2(\lambda_j-\lambda_i)x_0 & \text{otherwise}\end{cases}\\ &> 0.\end{aligned}$$

We also note that

$$\nabla l_2(y_1, y_2) = \begin{bmatrix} p_0 + (2\Lambda+\Gamma)y_1 + \Gamma y_2 \\ p_0 + \Gamma y_1 + (2\Lambda+\Gamma)y_2 \end{bmatrix},$$
$$\nabla e_2(y_1, y_2) = \begin{bmatrix} \Gamma x_0 - (2\Lambda-\Gamma)y_1 + \Gamma y_2 \\ \Gamma x_0 + \Gamma y_1 - (2\Lambda-\Gamma)y_2 \end{bmatrix}.$$

We now distinguish several subscenarios, and show how we can find an ϵ_2 in each case such that $[\epsilon_1'\epsilon_2']\nabla l_2(y^*_1, y^*_2) \leq 0$ and $[\epsilon_1'\epsilon_2']\nabla e_2(y^*_1, y^*_2) \geq 0$ in each case.

First, if $\epsilon_1'\Gamma(y^*_1+y^*_2) \leq 0$ and $\epsilon_1'\Gamma(x_0+y^*_1+y^*_2) \geq 0$, then $[\epsilon_1' 0]\nabla l_2(y^*_1, y^*_2) \leq 0$ and $[\epsilon_1' 0]\nabla e_2(y^*_1, y^*_2) \geq 0$.

Second, if $\epsilon_1'\Gamma(y^*_1+y^*_2) \leq 0$ but $\epsilon_1'\Gamma(x_0+y^*_1+y^*_2) < 0$, then let $\epsilon_2 = \epsilon_1$. Note that $\epsilon_1'\Gamma(x_0+y^*_1+y^*_2) < 0$ requires $y^*_{1,j}+y^*_{2,j} < y^*_{1,i}+y^*_{2,i}$, so $\epsilon_1'\Lambda(y^*_1+y^*_2) \leq 0$ and

$\epsilon_1'\Gamma(y_1^*+y_2^*)\le 0$, and hence $[\epsilon_1'\ \epsilon_2']\nabla l_2(y_1^*, y_2^*)\le 0$. In addition, we have

$$\begin{aligned}
&[\epsilon_1'\epsilon_2']\nabla e_2(y_1^*, y_2^*)/\epsilon\\
&\quad =2(\epsilon_1'\Gamma x_0-\epsilon_1'(2\Lambda-\Gamma)(y_1^*+y_2^*))/\epsilon\\
&\quad =2((\gamma_j-\gamma_i)x_0-(2\lambda_j-\gamma_j)(y_{1,j}^*+y_{2,j}^*)+(2\lambda_i-\gamma_i)(y_{1,i}^*+y_{2,i}^*))\\
&\quad \ge 2((\gamma_j-\gamma_i)x_0-((2\lambda_j-\lambda_i)-(\gamma_j-\gamma_i))(y_{1,j}^*+y_{2,j}^*))\\
&\quad \ge \begin{cases}2(\gamma_j-\gamma_i)x_0 & \text{if } 2(\lambda_j-\lambda_i)-(\gamma_j-\gamma_i)\ge 0,\\ 4(\lambda_j-\lambda_i)x_0 & \text{otherwise}\end{cases}\\
&\quad >0.
\end{aligned}$$

Finally, consider the case $\epsilon_1'\Gamma(y_1^*+y_2^*)\ge 0$. Then let $\epsilon_2=-\epsilon_1$ and note that

$$\begin{aligned}
\left[\epsilon_1'\ \epsilon_2'\right]\nabla l_2(y_1^*, y_2^*)/\epsilon &= 2\epsilon_1'\Lambda(y_1^*+y_2^*)/\epsilon\\
\left[\epsilon_1'\ \epsilon_2'\right]\nabla e_2(y_1^*, y_2^*)/\epsilon &= -2\epsilon_1\Lambda(y_1^*+y_2^*)/\epsilon,
\end{aligned}$$

and we therefore need to show that $\epsilon_1'\Lambda(y_1^*+y_2^*)\le 0$. We claim this is always true under the given conditions. Note that $\epsilon_1'\Gamma(y_1^*+y_2^*)\ge 0$ requires $y_{2,i}^*\le(\gamma_j/\gamma_i)y_{2,j}^*$. Therefore,

$$\begin{aligned}
\epsilon_1'\Lambda(y_1^*+y_2^*)/\epsilon &= (\lambda_j y_{1,j}^*-\lambda_i y_{1,i}^*)+(\lambda_j y_{2,j}^*-\lambda_i y_{2,i}^*)\\
&\le \lambda_j y_{2,j}^*-\lambda_i y_{2,i}^*\\
&\le \lambda_j y_{2,j}^*-\frac{\lambda_i\gamma_j}{\gamma_i}y_{2,j}^*\\
&< \lambda_j y_{2,j}^*-\lambda_j y_{2,j}^*\\
&= 0,
\end{aligned}$$

where in the first inequality, we use $0\ge y_{1,i}^*\ge y_{1,j}^*$ and $\lambda_j>\lambda_i$, in the second inequality we use $y_{2,i}^*\le(\gamma_j/\gamma_i)y_{2,j}^*$ and in the third inequality we use $\lambda_i/\gamma_i>\lambda_j/\gamma_j$.

Therefore, in each circumstance we have constructed a feasible direction for which l_1 and l_2 are no larger, e_2 is no smaller, and e_1 is strictly larger. Because $\pi<1$, this means we have found a direction that still satisfies all the problem constraints (no-short sales and margin constraints) with strictly larger objective, contradicting the optimality of (y_1^*, y_2^*).

Situation 2. Assume that at the optimal solution, the no-short sales constraints are binding for asset i but not binding for asset j (using an argument similar to the one above, we can argue that $y_{1,i}^*+y_{2,i}^*<y_{1,j}^*+y_{2,j}^*$ must hold, meaning it can never be the case that the constraints are binding in the reverse direction). Note that the no-short sales constraint is binding for asset i, so the above argument no longer applies (as we cannot sell any more of asset i).

For the next two cases, the following will be of use.

Lemma 1. *Let* $r_{z_1,z_2}\colon \mathbb{R}_+^2\to\mathbb{R}$ *be the family of functions*

$$r_{z_1,z_2}(\lambda,\gamma)\triangleq\frac{((1-\pi)+\rho_1 z_1)\gamma-2(\pi+(1+\rho_2)z_2)\lambda-z_1(p_0/x_0)}{(\pi-1+(1-\rho_1)z_1)\gamma+2(1+\pi+(1+\rho_1)z_1+2(1+\rho_2)z_2)\lambda},$$

parameterized by $(z_1, z_2)\in\mathbb{R}_+^2$, *and where* $\pi\in[0,1)$. *If* $\lambda_i<\lambda_j$, $\gamma_i<\gamma_j$, *and* $\gamma_i/\lambda_i<\gamma_j/\lambda_j$, *then for any* $0\le z_1<\infty$, $0\le z_2<\infty$, $r_{z_1,z_2}(\lambda_i,\gamma_i)<r_{z_1,z_2}(\lambda_j,\gamma_j)$.

Proof. We will fix a λ and a γ as well as all parameters in the function. The claim holds if and only if $\epsilon'\nabla r_{z_1,z_2}(\lambda,\gamma)>0$ for all vectors $\epsilon=[\epsilon_\lambda\ \epsilon_\gamma]'\in\mathbb{R}_{++}^2$ with $\epsilon_\lambda<(\lambda/\gamma)\epsilon_\gamma$ (i.e., moving in any directions such that λ and γ increase, as does their ratio γ/λ, must increase $r_{z_1,z_2}(\lambda,\gamma)$). After some algebra, we arrive at

$$\nabla r_{z_1,z_2}(\lambda,\gamma)\propto\begin{bmatrix}BD\gamma-A\left(C\gamma-\dfrac{z_1p_0}{x_0}\right)\\[2mm] AC\lambda-D\left(B\lambda-\dfrac{z_1p_0}{x_0}\right)\end{bmatrix},$$

where

$$\begin{aligned}
A&\triangleq 2(1+\pi+(1+\rho_1)z_1+2(1+\rho_2)z_2),\\
B&\triangleq -2(\pi+(1+\rho_2)z_2),\\
C&\triangleq 1-\pi+\rho_1 z_1,\\
D&\triangleq \pi-1+(1-\rho_1)z_1.
\end{aligned}$$

To verify that the gradient condition holds over all such ϵ, we need only check the extreme rays of the set, which are given by $\epsilon_1=[0\,1]'$ and $\epsilon_2=[(\lambda/\gamma)1]'$. For ϵ_1, this requires $AC\lambda-D(B\lambda-z_1p_0/x_0)>0$. If $B<0$, then $B\lambda-z_1p_0/x_0<0$, so, if $D\ge 0$, the condition $AC\lambda-D(B\lambda-z_1p_0/x_0)>0$ automatically holds. Otherwise, if $D<0$, we have

$$\begin{aligned}
AC\lambda-D(B\lambda-z_1p_0/x_0)&=(AC-BD)\lambda-D\frac{z_1p_0}{x_0}\\
&\ge (AC-BD)\lambda\\
&=(A+B)C\lambda-Bz_1\lambda\\
&\ge (A+B)C\lambda\\
&>0,
\end{aligned}$$

where we are using $B<0$, $D=-C+z_1$, $A>-B$, and $C>0$. For ϵ_2, we have

$$\begin{aligned}
\epsilon_2'\nabla r_{z_1,z_2}(\lambda,\gamma)&=\frac{\lambda}{\gamma}\left(DB\gamma-A\left(C\gamma-\frac{z_1p_0}{x_0}\right)\right)\\
&\quad+\left(AC\lambda-D\left(B\lambda-\frac{z_1p_0}{x_0}\right)\right)\\
&=\frac{\lambda}{\gamma}(BD-AC)\gamma+(AC-BD)\lambda+(A\lambda+B\gamma)\frac{z_1p_0}{x_0}\\
&=(A\lambda+B\gamma)\frac{z_1p_0}{x_0}\\
&\ge -B(2\lambda-\gamma)\frac{z_1p_0}{x_0}\\
&\ge 0,
\end{aligned}$$

where we are using $A\ge -2B$, $B<0$, $z_1\ge 0$, and the convexity condition $\lambda>\gamma/2$. Because any such ϵ in question must be a strictly positive combination of ϵ_1 and ϵ_2, the result now follows. □

Now note that in this case, $y_{1,i}^*=r_{z_1,z_2}(\lambda_i,\gamma_i)x_0$, with (z_1,z_2) the optimal Lagrange multipliers, because at the optimal solution the no-short sales constraint is active for asset i ($r_{z_1,z_2}(\lambda_i,\gamma_i)x_0$ is the form of the optimal solution, as discussed earlier, for asset i in this case). We will argue that for asset j, which satisfies $y_{1,j}^*+y_{2,j}^*>-x_0$ by assumption, we have $y^*_{1,j} > r_{z_1,z_2}(\lambda_j, \gamma_j)x_0$. Because Lemma 1 implies that $r_{z_1,z_2}(\lambda_j, \gamma_j) > r_{z_1,z_2}(\lambda_i, \gamma_i) = y^*_{1,i}/x_0$, this will establish the result.

We can interpret $r_{z_1,z_2}(\lambda_j, \gamma_j)x_0$ as the optimal first-period trade for asset j (with the Lagrange multipliers fixed at (z_1, z_2)) if we are forcing asset j to satisfy the no-short sales constraint tightly (i.e., forcing $y_{1,j} + y_{2,j} = -x_0$. Put another way, $r_{z_1,z_2}(\lambda_j, \gamma_j)x_0$ is the optimal solution y to the problem

$$\begin{aligned} &\text{maximize} \quad -\begin{bmatrix} y - y^*_{1,j} \\ -x - y^*_{2,j} - y \end{bmatrix}' \begin{bmatrix} f_j & h_j \\ h_j & g_j \end{bmatrix} \begin{bmatrix} y - y^*_{1,j} \\ -x - y^*_{2,j} - y \end{bmatrix} \\ &\text{subject to} \quad -x_0 \le y \le 0, \end{aligned}$$

where $f_j > g_j \ge h_j$. Assuming the inequalities are inactive at the optimal solution (it is easy to verify that $y = 0$ cannot be optimal, and if $-x_0$ is optimal, then $y = r_{z_1,z_2}(\lambda_j, \gamma_j)x_0 = -x_0 < y^*_{1,j}$, so we are done), the optimal solution y must satisfy the first-order condition:

$$(f_j - h_j)(y - y^*_{1,j}) - (g_j - h_j)(-x - y^*_{2,j} - y) = 0$$

$$\Updownarrow$$

$$(f_j + g_j - 2h_j)y = (f_j - h_j)y^*_{1,j} + (g_j - h_j)(-x - y^*_{2,j}).$$

Because $y^*_{1,j} + y^*_{2,j} > -x_0$ by assumption in this case, we have

$$(f_j + g_j - 2h_j)y < (f_j + g_j - 2h_j)y^*_{1,j},$$

and because $f_j > g_j \ge h_j$, this implies the optimal y must satisfy $y < y^*_{1,j}$, which gives us the result.

Situation 3. Assume that at the optimal solution, the no-short sales constraints are binding for both assets i and j. In this case, we have $y^*_{1,i} = r_{z_1,z_2}(\lambda_i, \gamma_i)x_0$ and $y^*_{1,j} = r_{z_1,z_2}(\lambda_j, \gamma_j)x_0$. The result now follows by Lemma 1.

Part (b): Throughout the proof for this result, we will use the notation (δ) to denote that a parameter in question (e.g., optimal solution, shadow price, etc.) is a function of the second-period shock size, $\delta \ge 0$, which will be varying. We will show that under the given conditions we can find a $\hat{\delta}$ with $z_1(\hat{\delta}) = 0$, the no-short sales constraints active for both asset i and asset j and $z_2(\hat{\delta})$ finite but arbitrarily large. We start with the following lemma.

Lemma 2. *Let* $r_{z_1,z_2}\colon \mathbb{R}^2_+ \to \mathbb{R}$ *be the family of functions as described in Lemma* 1. *Then if* $\gamma_i/\lambda_i < \gamma_j/\lambda_j$, *then there exists a* $0 \le z_2 < \infty$ *such that* $r_{0,z_2}(\lambda_i, \gamma_i) < r_{0,z_2}(\lambda_j, \gamma_j)$.

Proof. Note that when $z_1 = 0$, we can express the function in question as

$$r_{0,z_2}(\lambda, \gamma) = \frac{(1 - \pi - 2\pi(\lambda/\gamma)) - 2(1+\rho_2)(\lambda/\gamma)z_2}{(\pi - 1 + 2(1+\pi)(\lambda/\gamma)) + 4(1+\rho_2)(\lambda/\gamma)z_2}.$$

Denoting λ/γ by σ, note that we can write the functions as

$$r_{0,z_2}(\lambda_i, \gamma_i) = \frac{a_i - b_i z_2}{c_i + 2b_i z_2},$$

$$r_{0,z_2}(\lambda_j, \gamma_j) = \frac{a_j - b_j z_2}{c_j + 2b_j z_2},$$

where

$$a_i = 1 - \pi - 2\pi\sigma_i,$$

$$b_i = 2(1+\rho_2)\sigma_i,$$

$$c_i = \pi - 1 + 2(1+\pi)\sigma_i,$$

and analogously for (a_j, b_j, c_j). Because the denominators are both positive, one can verify that there exists a $z_2 \ge 0$ such that $r_{0,z_2}(\lambda_i, \gamma_i) < r_{0,z_2}(\lambda_j, \gamma_j)$ if

$$2(a_i b_j - a_j b_i) - (b_i c_j - b_j c_i) < 0$$

holds. This reduces to

$$2(a_i b_j - a_j b_i) - (b_i c_j - b_j c_i) = 2(1+\rho_2)(1-\pi)(\sigma_j - \sigma_i),$$

and because $\pi \in [0, 1)$, $\sigma_j < \sigma_i$ implies the result. □

Now consider the problem of finding the maximum possible $\bar{\delta} > 0$ such that the two-period problem is still feasible. Following the proof of Result 3, we find that such a $\bar{\delta}$ corresponds to the trade $y_1 = -x_0/2$, $y_2 = -x_0/2$, i.e., splitting up all assets equally across the two periods. Because the objective function for computing this $\bar{\delta}$ is strictly convex, the solution $y_1 = -x_0/2$ and $y_2 = -x_0/2$ can be the only solution that satisfies the second-period margin constraint at $\bar{\delta}$, and it must be that $\bar{\delta} > 0$ (because the trade $y_1 = -x_0/2$, $y_2 = 0$ satisfies the constraints for the problem $\delta = 0$ and the objective function for computing $\bar{\delta}$ is strictly convex). So, the feasible set to the original problem with $\delta = \bar{\delta}$ is a singleton at which the no-short sales constraints are obviously tight. By assumption, the first-period margin constraint is strictly satisfied, and hence $z_1(\bar{\delta}) = 0$. Moreover, for any $\delta > \bar{\delta}$, the problem is infeasible, and therefore we must have $z_2(\bar{\delta}) = +\infty$.

To complete the proof, we need to argue that we can find a small enough perturbation, $\epsilon > 0$, such that at $\bar{\delta} - \epsilon$, it is still optimal to have both box constraints active, $z_1(\bar{\delta} - \epsilon) = 0$, and $z_2(\bar{\delta} - \epsilon)$ is finite but arbitrarily large. If we can do this, we will have found a δ for which the first-period margin constraint is inactive, assets i and j are both tight on the no-short sales constraint, and $z_2(\delta)$ can be made as large as desired; because $y^*_{1,i}(\delta) = r_{0,z_2(\delta)}(\lambda_i, \gamma_i)x_0$ and $y^*_{1,j}(\delta) = r_{0,z_2(\delta)}(\lambda_j, \gamma_j)x_0$, the result will then follow by Lemma 2.

First, notice that the feasible set at $\delta = \bar{\delta}$ is a singleton, as argued above. For any $\epsilon > 0$, we are enlarging a single ellipsoid, and the feasible set must therefore still be compact (closed and bounded). The singleton at $\bar{\delta}$ is strictly contained inside the first-period margin ellipsoid. Because the feasible set shrinks to a singleton strictly contained in the first-period margin ellipsoid as $\epsilon \to 0$, we can find a sufficiently small $\epsilon_1 > 0$ such that the feasible set is still strictly contained inside the first-period margin constraint for all $\epsilon \in [0, \epsilon_1]$.

Now consider how z_2 varies with δ. Because z_2 is an optimal Lagrange multiplier, it is obtained by minimizing a rational function and therefore $z_2(\delta)$ is continuous, and $z_2(\bar{\delta}) = \infty$. Because it is a continuous function, we can find an $\epsilon_2 > 0$ such that $z_2(\bar{\delta} - \epsilon_2)$ is finite but arbitrarily large. Because it can be arbitrarily large, we can make it large enough such that the no-short sales constraints for both assets i and j must still be active for any $\epsilon \in [0, \epsilon_2]$.

Now take $\epsilon = \min(\epsilon_1, \epsilon_2) > 0$. We have $z_1(\bar{\delta} - \epsilon) = 0$, $z_2(\bar{\delta} - \epsilon) < \infty$, and both no-short sales constraints active at $\bar{\delta} - \epsilon$. The proof is complete. □

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